6.6
[W] 5-9 M/M/c

Balance-Gleichungen

$$\lambda p_n = (n+1)\mu\, p_{n+1} \quad \text{für } n = 0, 1, \dots, c-1$$

$$\lambda p_n = c\mu\, p_{n+1} \quad \text{für } n = c, c+1, \dots$$

$\rho := \frac{\lambda}{c\mu}$ (utilization factor). Auslastungsfaktor (Auslastung pro Server)

Für $n \geq c-1$ gilt: $p_{n+1} = \rho\, p_n$

Stabilität wenn $\rho < 1$ ist.

... (alternativ) wenn $\frac{\lambda}{\mu} < c$ ist

$\uparrow$
"offered load"

Auflösung der Balance-Gleichungen

$n \leq c$: $\quad p_n = (c\rho)^n \cdot \frac{1}{n!}\, p_0$

$n > c$: $\quad p_n = \rho^{n-c}\, p_c = \rho^{n-c} (c\rho)^c \cdot \frac{1}{c!}\, p_0$

$$= \rho^n \cdot \frac{c^c}{c!}\, p_0$$

Wegen $\sum_{n=0}^{\infty} p_n = 1$ ist

$$p_0 = \Big(\underbrace{\sum_{n=0}^{c-1} \frac{p_n}{p_0}}_{A} + \underbrace{\sum_{n=c}^{\infty} \frac{p_n}{p_0}}_{B} \Big)^{-1}$$

$$A = \sum_{n=0}^{c-1} (c\rho)^n \cdot \frac{1}{n!}$$

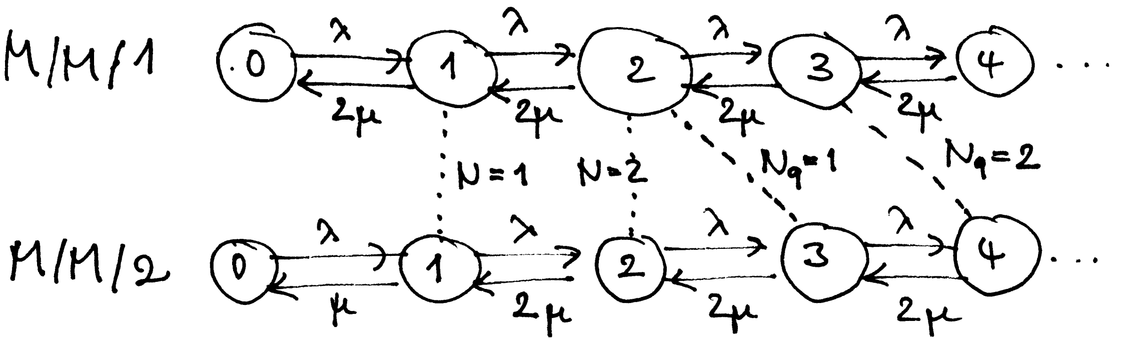
$$B = \sum_{n=c}^{\infty} \rho^n \frac{c^c}{c!} = \frac{c^c}{c!} \frac{\rho^c}{1-\rho} = \frac{(c\rho)^c}{c!} \cdot \frac{1}{1-\rho}$$

$$L = \sum_{n=0}^{\infty} n p_n = \dots$$

[W] Example 5-7: One Fast Versus Two Slow Servers

M/M/1 , service rate 2μ

M/M/2 , Service rate μ/channel



M/M/1

Balance-Gleichungen

$$\lambda p_0^{(1)} = 2\mu\, p_1^{(1)}$$

$$\lambda p_n^{(1)} = 2\mu\, p_{n+1}^{(1)} \quad (n \geq 0)$$

$$p_n^{(1)} = \left(\frac{\lambda}{2\mu}\right)^n p_0^{(1)} \quad (n \geq 0)$$

$$\frac{p_{n+1}^{(1)}}{p_n^{(1)}} = \frac{\lambda}{2\mu} \quad \text{für } n \geq 0.$$

M/M/2

Balance-Gleichungen

$$\lambda p_0^{(2)} = \mu\, p_1^{(2)}$$

$$\lambda p_n^{(2)} = 2\mu\, p_{n+1}^{(2)} \quad (n \geq 1)$$

$$p_n^{(2)} = \left(\frac{\lambda}{2\mu}\right)^{n-1}\left(\frac{\lambda}{\mu}\right) p_0^{(2)} \quad (n \geq 1)$$

$$= 2\left(\frac{\lambda}{2\mu}\right)^n p_0^{(2)}$$

$$\frac{p_{n+1}^{(2)}}{p_n^{(2)}} = \frac{\lambda}{2\mu} \quad \text{für } n \geq 1$$

$$\frac{p_1^{(2)}}{p_0^{(2)}} = \frac{\lambda}{\mu}$$

$$\frac{p_1^{(1)}}{p_0^{(1)}} < \frac{p_1^{(2)}}{p_0^{(2)}}$$

und $\cancel{\frac{p_n^{(2)}}{p_n^{(1)}} = 2\left(\frac{\lambda}{2\mu}\right)^n p_0^{(2)}}$

$$\sum_{n=0}^{\infty} p_n^{(1)} = p_0^{(1)} \sum_{n=0}^{\infty} \left(\frac{\lambda}{2\mu}\right)^n$$

$$= p_0^{(1)} \frac{1}{1-\frac{\lambda}{2\mu}} = 1 \quad \Rightarrow \quad p_0^{(1)} = 1 - \frac{\lambda}{2\mu}$$

$$\sum_{n=0}^{\infty} p_n^{(2)} = \sum_{n=1}^{\infty} 2\left(\frac{\lambda}{2\mu}\right)^n p_0^{(2)} + p_0^{(2)}$$

$$= p_0^{(2)} \frac{\cancel{2} \cdot \frac{\lambda}{\cancel{2}\mu}}{1-\frac{\lambda}{2\mu}} + p_0^{(2)}$$

$$= p_0^{(2)} \cdot \frac{\frac{\lambda}{\mu} + \left(1 - \frac{\lambda}{2\mu}\right)}{1-\frac{\lambda}{2\mu}}$$

$$= p_0^{(2)} \cdot \frac{1+\frac{\lambda}{2\mu}}{1-\frac{\lambda}{2\mu}} = 1 \quad \Rightarrow \quad p_0^{(2)} = \frac{1-\frac{\lambda}{2\mu}}{1+\frac{\lambda}{2\mu}}$$

$$\boxed{\frac{p_0^{(1)}}{p_0^{(2)}} = 1 + \frac{\lambda}{2\mu}}$$

Bch.: Bei Gleichgewicht ist $\boxed{p_{n+1}^{(2)} < p_n^{(1)} \text{ für } n = 1, 2, \ldots}$.

Bew.: $$\frac{p_{n+1}^{(2)}}{p_n^{(1)}} = \frac{2\left(\frac{\lambda}{2\mu}\right)^{n+1} p_0^{(2)}}{\left(\frac{\lambda}{2\mu}\right)^n p_0^{(1)}} = \frac{\frac{\lambda}{\mu} p_0^{(2)}}{p_0^{(1)}} = \frac{\frac{\lambda}{\mu}}{1+\frac{\lambda}{2\mu}} \underset{\Uparrow}{<} 1$$

Gleichgewicht $\Rightarrow \frac{\lambda}{2\mu} < 1 \Rightarrow \frac{\lambda}{\mu} < 1 + \frac{\lambda}{2\mu} \Rightarrow \frac{\frac{\lambda}{\mu}}{1+\frac{\lambda}{2\mu}} < 1$

Beh.: Bei Gleichgewicht ist $\boxed{p_n^{(2)} > p_n^{(1)} \quad \text{für } n \geq 1}$

Bew.: $$\frac{p_n^{(2)}}{p_n^{(1)}} = \frac{2\left(\frac{\lambda}{2\mu}\right)^n p_0^{(2)}}{\left(\frac{\lambda}{2\mu}\right) p_0^{(1)}} = \frac{2}{1+\underbrace{\frac{\lambda}{2\mu}}_{<1}} > 1.$$

Nun:

$$L_1 = \sum_{n=1}^{\infty} n p_n^{(1)} < \sum_{n=1}^{\infty} n p_n^{(2)} = L_2$$

$$Q_1 = \sum_{n=1}^{\infty} (n-1) p_n^{(1)} > \sum_{n=2}^{\infty} (n-2) p_n^{(2)} = Q_2$$

Interpretation?
