Aufgabe 1

2 Instrumente

Instr. 1 : $f_1(x) = 2x\,[0<x<1]$ ... X_1

Instr. 2 : $f_2(x) = 3x^2\,[0<x<1]$ ... X_2

ZV $Y \in \{1,2\}$: $P(Y=1) = P(Y=2) = \frac{1}{2}$

$$X = X_1\,[Y=1] + X_2\,[Y=2]$$

(a) Dichte von X?

$$P(X \le x) = P(X \le x \mid Y=1)P(Y=1) + P(X \le x \mid Y=2)P(Y=2)$$
$$= P(X_1 \le x)\cdot P(Y=1) + P(X_2 \le x)P(Y=2)$$

$$\frac{d}{dx} P(X \le x) = f_1(x)\,P(Y=1) + f_2(x)\,P(Y=2)$$
$$= \frac{1}{2} f_1(x) + \frac{1}{2} f_2(x)$$

(b) $P(Y=1 \mid X=\frac{1}{4}) = ?$

$$P(Y=1 \mid X=\tfrac{1}{4}) = \frac{P(Y=1, X=\frac{1}{4})}{P(X=\frac{1}{4})} = \frac{P(X=\frac{1}{4} \mid Y=1)P(Y=1)}{\sum_{i\in\{1,2\}} P(X=\frac{1}{4} \mid Y=i)P(Y=i)}$$

$$= \lim_{\varepsilon\to 0} \frac{\frac{1}{2\varepsilon}\int_{1/4-\varepsilon}^{1/4+\varepsilon} f_1(t)\,dt \cdot \cancel{\frac{1}{2}}}{\frac{1}{2\varepsilon}\int_{1/4-\varepsilon}^{1/4+\varepsilon} f_1(t)\,dt \cdot \cancel{\frac{1}{2}} + \frac{1}{2\varepsilon}\int_{1/4-\varepsilon}^{1/4+\varepsilon} f_2(t)\,dt \cdot \cancel{\frac{1}{2}}} = \frac{2\cdot\frac{1}{4}}{2\cdot\frac{1}{4} + 3\cdot\left(\frac{1}{4}\right)^2} = \frac{2}{2+\frac{3}{4}}$$

$$= \frac{8}{11}$$

Aufgabe 2

Z: rate at which customers are served in a queue.

$Z \sim$ p.d.f. $2e^{-2z}[0<z] = f(z)$

$T = \frac{1}{Z}$ (average waiting time)

$T \sim$ p.d.f. ?

$$P(T \leq t) = P\left(\frac{1}{Z} \leq t\right)$$

$$= P\left(\frac{1}{t} \leq Z\right)$$

$$= 1 - P\left(Z < \frac{1}{t}\right)$$

$$= 1 - \int_0^{1/t} f(z)\,dz$$

$$\frac{d}{dt} P(T \leq t) = -f\left(\frac{1}{t}\right) \cdot \left(-\frac{1}{t^2}\right)$$

$$= 2e^{-2 \cdot \frac{1}{t}}\left[0 < \frac{1}{t}\right] \cdot \frac{1}{t^2}$$

$$= \frac{2}{t^2} e^{-\frac{2}{t}}\,[0 < t]$$

Aufgabe 3

Seien X, Y unabhängig, identisch verteilt.

Dichte : $f(x) = e^{-x} [x > 0]$.

Seien $U = \dfrac{X}{X+Y}$, $V = X+Y$

Wie lauten die Dichten von U und V?

$$
\begin{aligned}
P(V \le v) &= P(X+Y \le v) \\
&= \int_0^v \underbrace{P(X+Y \le v \mid Y=y)}_{P(X \le v-y \mid Y=y)} f(y)\,dy \\
&= \int_0^v \int_0^{v-y} e^{-x}\,dx \; e^{-y}\,dy \\
&= \int_0^v \left(-e^{-x} \Big|_{x=0}^{x=v-y} \right) e^{-y}\,dy \\
&= \int_0^v \left(1 - e^{-(v-y)}\right) e^{-y}\,dy \\
&= \int_0^v e^{-y}\,dy - \int_0^v e^{-v}\,dy \\
&= 1 - e^{-v} - v e^{-v}
\end{aligned}
$$

$\frac{d}{dv} P(V \le v) = e^{-v} - e^{-v} + v e^{-v} = v e^{-v}$ Dichte von V

Aufgabe 4

$$U = \frac{X}{X+Y} \qquad U \in]0,1[$$

$$P(U \leq u) = P(X \leq u(X+Y))$$

$$P(X \leq u(X+Y) | X = c) = 1 - P(Y < c \cdot \tfrac{1-u}{u} | X = c)$$

$$P(U \leq u) = \int_0^\infty P(U \leq u | X = c) f_X(c) dc$$

$$= \int_0^\infty \left(1 - P(Y < c \cdot \tfrac{1-u}{u} | X = c)\right) e^{-c} dc$$

$$= 1 - \int_0^\infty P(Y < c \cdot \tfrac{1-u}{u} | X = c) e^{-c} dc$$

$$= 1 - \int_0^\infty \left(\int_0^{c \cdot \frac{1-u}{u}} e^{-y} dy \right) e^{-c} dc$$

$$= 1 - \int_0^\infty \left(1 - e^{-c \frac{1-u}{u}}\right) e^{-c} dc$$

$$= 1 - \left(1 - \int_0^\infty e^{-c \cdot \frac{1}{u}} dc\right)$$

$$= \int_0^\infty e^{-\frac{c}{u}} dc = -u e^{-\frac{c}{u}} \Big|_{c=0}^{c=\infty} = u$$

Gleichverteilung?

Aufgabe 5

Seien U, V unabhängig, identisch verteilt mit Dichte $f(x) = e^{-x}[x>0]$

Wie ist $\frac{U}{V}$ verteilt?

$$P\left(\frac{U}{V} \le z\right) = \iint_{\{\frac{u}{v} \le z\}} e^{-(u+v)} \, du \, dv$$

$$= \int_0^\infty \int_0^{zv} e^{-(u+v)} \, du \, dv$$

$$= \int_0^\infty e^{-v} \left(-e^{-u} \Big|_{u=0}^{u=zv} \right) dv$$

$$= \int_0^\infty e^{-v} \left(1 - e^{-zv}\right) dv$$

$$= \dots = 1 - \int_0^\infty e^{-v(1+z)} \, dv$$

$$= 1 + \frac{1}{1+z} e^{-v(1+z)} \Big|_{v=0}^{v=\infty}$$

$$= 1 - \frac{1}{1+z} = \frac{z}{1+z}$$

$$\frac{d}{dz} P\left(\frac{U}{V} \le z\right) = \frac{1+z-z}{(1+z)^2} = \frac{1}{(1+z)^2}$$