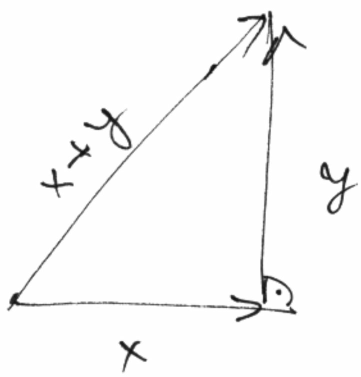
Lineare Algebra

Lektion 14

Orthogonalität von Vektoren



Pythagoras:

$$\boxed{\|x\|^2 + \|y\|^2 = \|x+y\|^2 \qquad (*)}$$

$$\|x\|^2 = x_1^2 + x_2^2 + \dots + x_n^2 \quad \text{(ebenfalls Pythagoras!)}$$
$$= x^T x$$

$$(*) \iff \underbrace{(x+y)^T(x+y)}_{x^T x + \underbrace{x^T y + y^T x}_{2x^T y} + y^T y} = x^T x + y^T y$$

$$\iff \boxed{x^T y = 0}$$

(B) $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $y = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$. $x+y = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$

$x^T y = 0$, $\|x\|^2 = 14$, $\|y\|^2 = 5$, $\|x+y\|^2 = 19$

Orthogonalität von Unterräumen

Seien S, T Unterräume.

$$S \perp T \quad :\Leftrightarrow \quad \forall_{\substack{v \in S \\ w \in T}} \; v \perp w$$

Anti-(B) S x,y-Ebene, T x,z-Ebene

$S \perp T$? nein!

denn es gilt z.B. nicht $\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\in S \cap T} \perp \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\in S \cap T}$

(B) $C(A^T) \perp N(A)$

$$\overset{\Downarrow}{v} = A^T y, \quad A\overset{\Downarrow}{w} = 0$$

$$v^T w = y^T \underbrace{Aw}_{0} = 0$$

(B) $C(A) \perp N(A^T)$

(B)

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix}$$

$$C(A^T) = \left\{ c \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \,\middle|\, c \in \mathbb{R} \right\}$$

$\dim C(A^T) = 1$

$\Rightarrow \dim N(A) = 2$

$$N(A) \perp \left\{ c \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \,\middle|\, c \in \mathbb{R} \right\}$$

$C(A^T) \perp N(A)$ und $\dim C(A^T) + \dim N(A) = n$

$C(A) \perp N(A^T)$ und $\dim C(A) + \dim N(A^T) = m$

"orthogonale Komplemente"

$\boxed{N(A^TA) = N(A)}$

denn: $x \in N(A) \Rightarrow Ax = 0 \Rightarrow A^TAx = 0 \Rightarrow x \in N(A^TA)$

und $x \in N(A^TA) \Rightarrow A^TAx = 0 \Rightarrow x^TA^TAx = 0 \Rightarrow Ax = 0 \Rightarrow x \in N(A$

<u>Anwendung</u>: $\left.\begin{array}{l} A \; m \times n\text{-Matrix} \\ m > n \\ \operatorname{rank}(A) = n \end{array}\right\} \Rightarrow A^TA$ invertierbar