1 Solving Hamiltonians

- Today we will study Dynamic Programming. In particular we will learn how to solve so called Hamiltonians. We do this to facilitate the understanding of chapter 4 in the book.

- Our strategy is to go through some relatively simple similar examples.

- A book I recommend on this is Barro and Sala-I-Martin, "Economic Growth".

- Mathematicians have long worried about dynamic problems.

- The first one to solve this kind of problem was probably Bernoulli in 1696. The theories are mostly used in physics but also economists have much use of them.

- We will deal with infinitely lived households that choose consumption and saving to maximize their utility subject to an intertemporal budget constraint (The so called Ramsey Model). Our method will be the following:

- First, to set up the households' problem.

- Second, to solve the model, looking for steady state values of consumption and the capital stock.

- Third, to show, in a so called phase diagram, how consumption and the capital stock change over time as they approach steady state.

- Fourth, to do some comparative statics to get a better feeling for the model.

1.1 Example 1

Consider first the problem

$$\max_{c_t} \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \tag{1}$$

$$s.t. \quad \begin{cases} \dot{k}_t = A k_t^\theta - c_t - \delta k_t \\ k_0 \ given \end{cases} \tag{2}$$

and

$$\lim_{t \to \infty} e^{-\rho t} \mu_t k_t = 0.$$

- c_t is consumption, ρ is the time discount rate, δ is the depreciation rate of physical capital, and k_t is the capital stock.

- $A > 0$ and γ (capturing the level of risk aversion), ρ, δ and θ are between zero and one.

- A positive ρ means that utils are valued less the later they are received.

- c_t is called a control variable.

- The constraints are dynamic in that they describe the evolution of the state of the economy as represented of the state variable k_t.

- The first constraint shows how the choice of the control variable, c_t, translates into a pattern of movement for the state variable $k_{t.}$

- This equation is called the cannonical equation or the law of motion.

- We note that both the control variable and the state variable are functions of time.

- The restriction k_0 says that the state variable k_t begins at a given value, i.e., k_0.

- The constraint $\lim_{t \to \infty} e^{-\rho t} \mu_t k_t = 0$ is called the transversality condition. It says that the chosen value of the state variable at the end of the planning horizon discounted at the rate ρ must be zero.

- The intuitive meaning of the transversality condition is "Since you can't leave any debt behind, and there is no point in leaving assets behind, set the net worth of your assets to zero".

- To put it differently, households would want to die with a debt (negative k_t) but this is not allowed. In addition, the value of the capital stock must be zero asymptotically, otherwise something valuable would be left over.

- The current value Hamiltonian, H, is

$$H = \frac{c_t^{1-\gamma}}{1-\gamma} + \mu_t \left(A k_t^{\theta} - c_t - \delta k_t \right) \qquad (3)$$

where μ_t is the current value costate (or shadow value) and $\mu_t = e^{\rho t} \lambda_t$.

μ_t represents the value of an increment of income received at time t in units of utils at time t.

λ_t (the present value costate) represents the value of an increment of income received at time t in units of utils at time 0.

In general, the first-order conditions are

$$H_c = 0, \qquad (4)$$

and

$$H_k = -\dot{\mu}_t + \rho \mu_t. \qquad (5)$$

So in our case we get

$$
\begin{aligned}
H_c &= 0 \Leftrightarrow c_t^{-\gamma} = \mu_t &(6)\\
H_k &= -\dot{\mu}_t + \rho\mu_t \Leftrightarrow &(7)\\
\mu_t\left(\theta A k_t^{\theta-1} - \delta\right) &= -\dot{\mu}_t + \rho\mu_t &(8)
\end{aligned}
$$

In search for the steady state solution we need to achieve a system of the two differential equations $\dot{k}_t(k_t, c_t)$ and $\dot{c}_t(k_t, c_t)$.

We already have a differential equation $\dot{k}_t(k_t, c_t)$. What we need now is a differential equation $\dot{c}_t(k_t, c_t)$.

We also want to get rid of μ_t and $\dot{\mu}_t$ in the H_k-equation. To obtain an expression for $\dot{c}_t$ use the trick to differentiate μ_t with respect to time.

Use the fact that

$$H_c = 0 \Leftrightarrow c_t^{-\gamma} = \mu_t, \tag{9}$$

to achieve

$$-\gamma \dot{c}_t c_t^{-\gamma-1} = \dot{\mu}_t \tag{10}$$

Now, substitute for μ_t and $\dot{\mu}_t$ in the cannonical equation.

$$c_t^{-\gamma}\left(\theta A k_t^{\theta-1} - \delta\right) = -(-\gamma \dot{c}_t c_t^{-\gamma-1}) + \rho c_t^{-\gamma} \tag{11}$$

From here we get the differential equation $\dot{c}_t(k_t, c_t)$.

$$\dot{c}_t = (A k_t^{\theta-1} - \delta - \rho)\frac{c_t}{\gamma} \tag{12}$$

So we have the following non-linear system

$$\begin{cases} \dot{k}_t = A k_t^{\theta} - c_t - \delta k_t \\ \dot{c}_t = \frac{c_t}{\gamma}\left(\theta A k_t^{\theta-1} - \rho - \delta\right) \end{cases} \tag{13}$$

We now look for the steady state values of k and c, In steady state both $\dot{k} = 0$ and $\dot{c} = 0$.

Using the previous system of differential equations we get

$$\dot{k}_t = 0 \Rightarrow c_t = Ak_t^{\theta} - \delta k_t, \tag{14}$$

and

$$\dot{c}_t = 0 \Rightarrow \theta A k_t^{\theta - 1} - \rho - \delta = 0 \Leftrightarrow k_t = \left(\frac{\theta A}{\rho + \delta}\right)^{\frac{1}{1-\theta}}. \tag{15}$$

The steady state is given by the k^* and c^*, which solve these two equations simultaneously.

Hence, the steady state values are

$$k^* = \left(\frac{\theta A}{\rho + \delta}\right)^{\frac{1}{1-\theta}} \tag{16}$$

and

$$c^* = A\left(\frac{\theta A}{\rho + \delta}\right)^{\frac{\theta}{1-\theta}} - \delta\left(\frac{\theta A}{\rho + \delta}\right)^{\frac{1}{1-\theta}} \tag{17}$$

- We can now make a phase diagram to describe the dynamics around the steady state.

- Here comes the method:

- First draw the curves given by the $\dot{k}_t = 0$ and $\dot{c}_t = 0$ equations. They were given by

$$c_t = Ak_t^{\theta} - \delta k_t \tag{18}$$

and

$$k_t = \left(\frac{\theta A}{\rho + \delta}\right)^{\frac{1}{1-\theta}}. \tag{19}$$

- To understand the direction of motion over time, we now want to know what happens when we aren't on these lines.

- Let's look at

$$\begin{cases} \dot{k}_t = Ak_t^{\theta} - c_t - \delta k_t \\ \dot{c}_t = \frac{c_t}{\gamma}\left(\theta A k_t^{\theta-1} - \rho - \delta\right) \end{cases} \tag{20}$$

  for a while.

- Assume we are on the $\dot{k}_t = 0$ curve, and that we increase consumption. Then we see that $\dot{k}_t$ becomes negative. So, above the $\dot{k}_t = 0$ curve, capital is decreasing which we illustrate by left arrows in the diagram.

- If we instead decrease consumption compared to our initial point on $\dot{k}_t = 0$ the contrary is true, i.e., capital is increasing (illustrated by right arrows).

- Now consider an initial point on the curve $\dot{c}_t = 0$. If we increase k then $\dot{c}_t$ becomes negative. This is illustrated by the down arrows to the right of $\dot{c}_t = 0$.

- Analogously, if we here decrease k, then $\dot{c}_t$ becomes positive, which is illustrated by the upward arrows.

- We can now find the saddle path, which shows the economy's way to the equilibrium.

- If the households would start off that curve, they would diverge away from the equilibrium.

- What happens to the economy if for example the discount factor increases unexpectedly and permanently (i.e., if ρ increases)?

- The $\dot{c}_t = 0$ shifts inward. The former steady state is now on an explosive trajectory. The only solution for not floating off into infinity is to jump to the new saddle path. We do only jump vertically and change consumption (destruction of capital is not optimal).

- Hence, we immediately jump up to the new saddle path and stay on the saddle path. We will now follow the saddle path towards the new steady state.

- The response is therefore to immediately increase consumption. The steady state is characterized by a smaller capital stock and less consumption than before.

- The intuition is clear. When ρ increases, we become more impatient, consume much early and have a smaller capital stock in steady state.

- What happens if the technology unexpectedly improves (if A increases).

- $\dot{k}_t = 0$ curve turns outwards and $\dot{c}_t = 0$ shifts outwards.

- We jump upwards to a new saddle path such that households immediately consume more. The capital stock cannot however increase discontinuously. The steady state is characterized by more consumption and a larger capital stock.

Before finishing this example we want to know if the necessary conditions for a solution also are sufficient.

To check this, we study the so called Hessian, which consists of the second derivatives

$$D^2 f(c_t, k_t) = \begin{matrix} H^c_{cc} & H^c_{kc} \\ H^c_{ck} & H^c_{kk} \end{matrix} \tag{21}$$

If $H^c_{cc} < 0$ and the determinant of the Hessian is positive, then these conditions are sufficient for a solution.

The first-order conditions were

$$\begin{aligned} H_c &= 0 \Leftrightarrow c_t^{-\gamma} = \mu_t & (22) \\ H_k &= -\dot{\mu}_t + \rho\mu_t \Leftrightarrow & (23) \\ \mu_t \left(\theta A k_t^{\theta-1} - \delta\right) &= -\dot{\mu}_t + \rho\mu_t & (24) \end{aligned}$$

Let's calculate the second derivatives

$$H^c_{cc} = -\gamma c_t^{-\gamma-1} \quad (25)$$
$$H^c_{kk} = (\theta-1)\mu_t \theta k_t^{\theta-2} \quad (26)$$
$$H^c_{ck} = 0 \quad (27)$$

The determinant is given by $H^c_{cc}H^c_{kk} - H^c_{ck}H^c_{kc}$, which in this case is

$$-\gamma c_t^{-(1+\gamma)}(\theta-1)\mu_t\theta k_t^{\theta-2} > 0 \text{ for } \gamma > 0 \text{ and } \theta < 1.$$

Therefore, the necessary conditions are also sufficient when the utility function and the production functions are concave.

A linear function, when $\gamma = 0$, does for example not work.

1.2 The typical problem

- The problem is given by:

$$\max_{c_t} V_0 = \int_0^\infty v[k_t, c_t, t]dt \tag{28}$$

$$s.t. \quad \begin{cases} \dot{k}_t = g[k_t, c_t, t] \\ k_0 = k_0 > 0, given \end{cases} \tag{29}$$

and

$$\lim_{t \to \infty} e^{-\rho t} \mu_t k_t = 0 \tag{30}$$

- The agents choose, or control, a number of variables called control variables so as to maximize an objective function subject to some constraints.

- These constraints are dynamic in that they describe the evolution of the state of the economy as represented by a set of state variables over time.

- $\dot{k}_t = g[k_t, c_t, t]$, is a differential equation in k_t which shows how the choice of the control variable, c_t, translates into a pattern of movement for the state variable k_t.

1.3 Example 2

Consider the following problem

$$\max_{c_t} \int_0^\infty e^{-\rho t} \ln c_t dt \tag{31}$$

$$s.t. \quad \begin{cases} \dot{k}_t = k^\alpha - c_t \\ k(0) \text{ given} \end{cases}$$

and the transversality condition (no debt when dying).

Set up the current value Hamiltonian

$$H^c = \ln c_t + \mu_t(k^\alpha - c_t) \tag{32}$$

where $\mu_t = e^{\rho t}\lambda_t$.

Note, there is one effect from consumption on utility and then one effect from consumption on the transition of the capital stock.

The necessary conditions for a solution are

$$\begin{aligned} H^c_c &= \frac{1}{c_t} - \mu_t = 0 & (33)\\ H^c_k &= \rho\mu_t - \dot{\mu}_t \Leftrightarrow & (34)\\ \mu_t \alpha k^{\alpha-1} &= \rho\mu_t - \dot{\mu}_t. & (35) \end{aligned}$$

Are the necessary conditions sufficient? Let's check the Hessian again

$$D^2 f(c_t, k_t) = \begin{matrix} H^c_{cc} & H^c_{kc} \\ H^c_{ck} & H^c_{kk} \end{matrix} \tag{36}$$

If $H^c_{cc} < 0$ and the determinant of the Hessian is positive, then these conditions are sufficient for a solution.

The second derivatives are

$$H^c_{cc} = -\frac{1}{c_t^2} \tag{37}$$

$$H^c_{kk} = (\alpha - 1)\mu_t \alpha k_t^{\alpha-2} \tag{38}$$

$$H^c_{ck} = 0 \tag{39}$$

The determinant is given by $H^c_{cc}H^c_{kk} - H^c_{ck}H^c_{kc}$, which in this case is

$$-\frac{1}{c_t^2}(\alpha - 1)\mu_t \alpha k_t^{\alpha-2} > 0.$$

Because the logarithmic consumption function and the production function are concave, the conditions are sufficient for a solution.

Again, to find the steady state solution we need to achieve a system of the two differential equations $\dot{k}_t(k_t, c_t)$ and $\dot{c}_t(k_t, c_t)$.

To find $\dot{c}_t$, we again use the trick to differentiate μ_t with respect to time. We then get $\mu_t = \frac{1}{c_t}$ to get $\dot{\mu}_t = -\frac{1}{c_t^2}\dot{c}_t$.

Substituting this into $\mu_t \alpha k^{\alpha-1} = \rho\mu_t - \dot{\mu}_t$, yields

$$\alpha\frac{1}{c_t}k^{\alpha-1} = \rho\frac{1}{c_t} + \frac{1}{c_t^2}\dot{c}_t \tag{40}$$

Since the equation for $\dot{k}_t$ is given by the constraint we have the following system of equations

$$\begin{cases} \dot{k}_t = k^\alpha - c_t \\ \dot{c}_t = \alpha k_t^{\alpha-1} c_t - \rho c_t \end{cases} \tag{41}$$

From here we can write a nice phase diagram showing the dynamics of the model.

In steady state both $\dot{k} = 0$ and $\dot{c} = 0$. We therefore get

$$\dot{c}_t = 0 \Leftrightarrow \alpha k_t^{\alpha-1} c_t - \rho c_t = 0 \Leftrightarrow k_t = (\frac{\rho}{\alpha})^{\frac{1}{\alpha-1}} \quad (42)$$

$$\dot{k}_t = 0 \Leftrightarrow c_t = k_t^{\alpha} \quad (43)$$

$$k^* = (\frac{\rho}{\alpha})^{\frac{1}{\alpha-1}} = (\frac{\alpha}{\rho})^{\frac{1}{1-\alpha}} \quad (44)$$

$$c^* = k^{\alpha} = (\frac{\alpha}{\rho})^{\frac{\alpha}{1-\alpha}} \quad (45)$$

What is the economics?

In steady state, everything is consumed while nothing is invested.

What if α increases. Then the curve $\dot{k}_t$ turns upwards and $\dot{c}_t$ shifts outwards. In the new equilibrium there is more investments and more consumption compared to before.

Assume now that $\alpha = \frac{1}{2}$. Then the solution boils down to.

$$k^* = (\frac{1}{2\rho})^2 \tag{46}$$

$$c^* = \frac{1}{2\rho} \tag{47}$$

It is clear that a higher impatience means that much of the capital stock is eaten up and steady-state consumption must be low.

1.4 Example 3

Consider the following problem

$$\max_{c_t} \int_0^\infty e^{-\rho t} \ln c_t dt \tag{48}$$

$$s.t. \quad \begin{cases} \dot{k}_t = k_t^\alpha - \pi k_t - c_t \\ k(0) \text{ given} \end{cases}$$

where π is the depreciation rate of physical capital, k_t is the capital stock and c_t is consumption. A, π and α are constants between zero and one.

The current value Hamiltonian is

$$H^c = \ln c_t + \mu_t(k_t^\alpha - \pi k_t - c_t) \tag{49}$$

The necessary conditions for a solution are

$$\frac{\partial H^c}{\partial c_t} = 0 \tag{50}$$
$$H_k^c = \rho\mu_t - \dot{\mu}_t \tag{51}$$

In this case, we get

$$\frac{\partial H^c}{\partial c_t} = \frac{1}{c_t} - \mu_t = 0 \tag{52}$$
$$H_k^c = (\alpha k_t^{\alpha-1} - \pi)\mu_t = \rho\mu_t - \dot{\mu}_t \tag{53}$$

We take the time derivative in the first equation to get $\dot{\mu} = -\frac{1}{c_t^2}\dot{c}$. Again we use this fact to substitute for in the cannonical equation with the purpose of getting $\dot{c}_t(k_t, c_t)$

Hence,

$$(\alpha k^{\alpha-1} - \pi)\frac{1}{c_t} = \rho\frac{1}{c_t} + \frac{1}{c_t^2}\dot{c} \tag{54}$$

Solve for $\dot{c}$

$$\dot{c} = (\alpha k_t^{\alpha-1} - \pi - \rho)c_t \tag{55}$$

Hence, we achieve a very similar result as in the first model above.

$$\begin{cases} \dot{k}_t = k_t^{\alpha} - \pi k_t - c_t \\ \dot{c}_t = (\alpha k_t^{\alpha-1} - \pi - \rho)c_t \end{cases} \tag{56}$$

We can now draw the phase diagram.

Finally, we can solve for the steady state values of consumption and the capital stock:

$$\begin{aligned} \dot{c} &= 0 \leftrightarrow (\alpha k_t^{\alpha-1} - \pi - \rho)c_t = 0 \leftrightarrow & (57) \\ k^* &= (\frac{\alpha}{\pi + \rho})^{\frac{1}{1-\alpha}} & (58) \end{aligned}$$

$$\dot{k}_t = 0 \leftrightarrow c_t^* = (\frac{\alpha}{\pi + \rho})^{\frac{\alpha}{1-\alpha}} - \pi(\frac{\alpha}{\pi + \rho})^{\frac{1}{1-\alpha}} \tag{59}$$